Productivity Revolutions and Science Driven Growth

Phillip Garner

Brigham Young University

Abstract
The advent of modern science made possible the emergence of sustained economic growth. Without growth in scientific knowledge, the productivity growth experienced during the Industrial Revolution would have eventually diminished, as did growth from previous ‘productivity revolutions’ (i.e. agriculture, expansions of trade). Yet scientific knowledge, as distinct from technology or economic productivity, is often not explicitly included in models of long run growth. This paper presents a simple model that, by making this distinction, allows for more realistic modeling of the emergence of modern sustained growth by highlighting the importance of the scientific revolution that preceded and coincided with it.

JEL Classifications: O3, O4
Keywords: Economic Growth, Science, Technology, Productivity

---

1 Correspondence information: Phillip Garner, Department of Economics, Brigham Young University, Provo, Utah 84602. Email: pgarner@byu.edu
1. Introduction

The emergence of sustained modern growth coincided with an enormous expansion of scientific knowledge. The productivity growth experienced during the Industrial Revolution was able to continue to the present because of the host of technological innovations made possible by advances in scientific understanding. For example, a mathematical understanding of electromagnetism is necessary for the widespread generation, transmission, and use of electricity. Technology can also have an effect on scientific progress: current climate modeling depends on the enormous amount of computing power made possible by technological innovations like the microprocessor.

Despite the importance of growth in scientific knowledge to sustainable economic growth, it is seldom explicitly included, as distinct from technology, in models of long-run growth. Is it useful for long-run growth models to distinguish between productivity growth and growth in scientific knowledge? This paper argues that it is. It can provide a more realistic picture of how, and why, modern economic growth emerged in the Western Europe. It also places some restrictions on the growth that was possible in previous societies.

Mokyr (2002, 2005) argues persuasively on the importance of increasing scientific knowledge and the Enlightenment to the emergence of sustained economic growth. He argues that the Enlightenment culture of seeking to expand and disseminate useful knowledge helped create a conducive environment for an Industrial Revolution to occur. In addition, even though in a strict sense modern science did not contribute directly to the major technological innovations of the early years of the Industrial Revolution, science did provide the foundation for the sustained growth that followed, particularly in the latter part of the 19th century and beyond.

This paper presents a simple model of productivity revolutions and science driven growth. How modern science emerged in Europe\(^2\), and the feedback between science, technology, and the economy is very complex and this paper is not ambitious enough to try to make the growth of scientific knowledge entirely endogenous (although some possibilities along these lines are discussed in section 5). Instead, assuming growth in scientific knowledge to be

\(^2\) Of course, the development of modern science in Europe did not occur in isolation, and was dependent on significant imports of knowledge from other regions. In particular, the transmission, and elucidation, of Classical texts via the Islamic world, and the adoption of ‘Arabic’ (Indian in origin) numerals through the same source, both played important roles in the intellectual development of Europe during the Middle Ages.
exogenous, it addresses a somewhat more modest set of questions. How does productivity, population, human capital, and economic output respond to the presence of (or lack of) growth in scientific knowledge? How would ‘productivity revolutions’, like the advent of agriculture, expansions of trade, or something like the early Industrial Revolution, play out if there is no corresponding growth in scientific knowledge?

A model of science driven growth must come to terms with the historical fact that substantial technological and economic progress has been made in the past, and in the pre-modern era this progress was most often independent of the state of scientific knowledge at the time. Simply assuming that, in the absence of growth in scientific knowledge, productivity growth is non-existent or monotonically decreasing over time would be misleading and historically inaccurate. But the opposite approach, that productivity growth is monotonically accelerating over time, perhaps due to scale effects, misses the qualitative difference between science driven growth and growth in the pre-modern era. There is no a priori reason to believe that technological advances like those made during the early years of the Industrial Revolution must necessarily coincide with advances in scientific knowledge, or that the emergence of, and sustained growth in, formal science is inevitable. And without expanding scientific knowledge, and the understanding of the natural world which it offers, sustained productivity growth over the very long run is not possible.

With this in mind, there are two underlying ideas that guide the set-up of the model. First, that growth in scientific knowledge is necessary for sustained productivity growth over the very long run. Second, that substantial productivity growth is possible, and recurrent, even in there is no growth in scientific knowledge, but this type of growth dissipates over time and is essentially limited. This leads to the distinction between two types of innovations: incremental and major. Incremental innovations represent the regular process of refining and improving upon production processes. Major innovations, which could be considered as general purpose technologies, represent significant technological discoveries that open up a host of new ways of producing output. The arrival of a major innovation is what is meant by a ‘productivity revolution’.

3 Speaking of the scientific revolution and the technology it made possible Mokyr (2002, p286) writes, “There was nothing inevitable about all of this, and it is far from obvious that, had western Europe never existed, or had it been wiped out by Genghis Khan, that some other society would have eventually developed X rays, freezer-dried coffee, and solar powered desk calculators”.

4 Thousands of years of human civilization passed before a lack of modern science could have become a binding constraint on economic growth (in the second half of the 19th century or so), but this long length of time reflects a slow average growth rate of productivity.
Improving crop yields through better types and applications of fertilizers would be an example of incremental innovation. The discovery of agriculture itself, or a significant set of new crops, would be examples of major innovations. Incremental innovations arrive in a continuous fashion where the growth rate of productivity due to incremental innovations is assumed to depend positively on the size of the population and the quality of human capital embodied in the population, and negatively on the current level of productivity; a ‘fishing-out’ effect of useful ideas. Major innovations arrive in a discrete fashion, and by expanding the set of useful ideas, sets back these fishing-out effects, resulting in temporarily higher productivity growth.

An increase in the amount of scientific knowledge available in society is assumed to have two direct effects on productivity. First, it raises (in equilibrium) the quality of the human capital embodied in the labor force, thus increasing productivity growth due to incremental innovations. Second, it can make more major innovations accessible (for example, the scientific understanding of electromagnetism due to Faraday, Maxwell and others made possible the major innovation of electrical power) and also increase the arrival rate of major innovations.5

It is assumed that for a given level of scientific knowledge the number of accessible (already discovered and yet to be discovered) major innovations is finite and that if the level of scientific knowledge increases sufficiently then new major innovations become accessible. For example, before the advent of modern science, computer-based information technology was not accessible, but agriculture, basic metallurgy, and possibly steam power were. In addition it is assumed that, as long as some major innovations are accessible and as yet undiscovered, that the arrival rate of major innovations is increasing in the population size and the quality of human capital.6

Population growth will depend positively on productivity growth, by relaxing a Malthusian-type of constraint (up to a limit). Population growth will also depend negatively on

---

5 These assumptions in turn implicitly assume that there exist economic incentives to both acquire additional knowledge and to use that knowledge to create new technology. This may not be the case, of course, if the economic institutional environment does not reward technological innovation, for example if there is not effective protection against expropriation or if there are barriers to entry of new firms and competition. Within the model this could be captured by a small response of human capital acquisition to increases in scientific knowledge, and a small effect of human capital on the rate of incremental innovation and the arrival rate of major innovations.

6 There could also be dependency among major innovations, so that the accessibility of some major innovations is dependent on other major innovations already being discovered. For example, the major innovation of electrical power is necessary for the major innovation of electronic computing to be accessible.
the growth rate of human capital, capturing a trade-off parents face in terms of resources between
the quantity and quality of their offspring.

Output is produced by combining labor and technology (as quantified in the level of
productivity). Holding productivity constant, there are decreasing returns to labor. That is, there
is a fixed (but renewable) resource that is necessary for production, which can represent land or
natural resources more generally. For simplicity, there is no physical capital accumulation in the
model. In the presence of a diminishing marginal product of physical capital, steady state growth
will be driven by productivity growth, so abstracting from physical capital in a very long run
growth model is probably not too limiting.

With a few additional parameter assumptions, the model works as follows: in the absence
of growth in scientific knowledge, productivity growth is positive only after the arrival of a
major innovation, then over time falls back to zero. As the population grows following major
innovations, both the arrival rate and the initial impact of major innovations increase over time,
leading to rising ‘average’ productivity and population growth. There is no growth in human
capital. Given a fixed amount of scientific knowledge, the number of accessible major
innovations is limited, so eventually population and productivity growth comes to a halt.

With a positive growth rate of scientific knowledge, productivity growth due to
incremental innovation can be positive in the steady state7, and new major innovations become
accessible (eventually). Productivity growth is still uneven over time, depending partially on the
random arrival of major innovations, but it will trend about a positive number, rather than zero as
in the case above. The growth rate of the quality of human capital is positive. The population
growth rate will be increasing in the growth rate of scientific knowledge for relatively low
growth rates of scientific knowledge, and will be decreasing for relatively high growth rates.

The Industrial Revolution, a demographic transition, and the emergence of sustained
growth could be captured in the model in two ways. First, there could be the more or less
simultaneous (and coincidental) arrival of a major innovation (steam power and resulting
mechanization of some production) and an acceleration in the growth rate of scientific
knowledge. Alternatively, there could be an earlier increase in the growth rate of scientific
knowledge, beginning say in the 17th century, which led to somewhat higher human capital

---

7 This assumption is not entirely essential to the model. It could be that long-run productivity growth is completely
driven by the arrival of major innovations, and growth in scientific knowledge makes more of these accessible over
time, thus allowing growth on average to be greater than zero.
growth, which increased the likelihood of something like the Industrial Revolution arriving. Further increases in the growth rate of scientific knowledge would fuel income per capita growth and eventually lead to a demographic transition.

The rest of the paper is organized as follows. Related literature and the place of this paper within that literature are discussed in section 2. Section 3 presents the basic model. Section 4 applies the model to two cases: long run productivity growth with no growth in scientific knowledge, and the science driven emergence of sustained growth. Section 5 briefly examines the potential roles population levels, human capital, and technology can play in the advancement of scientific knowledge. Section 6 concludes. Extensions of the basic model are presented in the appendix.

2. Related Literature

Recently there has been increased interest in questions of economic growth over the very long run. How did the world escape from centuries of Malthusian (relative) stagnation and transition to modern growth? What roles did evolution, population size, human capital investments, and the demographic transition play? Was, as Jones (2001) asks, something like the Industrial Revolution inevitable? Great progress has been made in developing unified models of economic growth that capture the entirety of the growth process, from primitive economies to the sustained, technologically driven growth that characterizes modern economies. A common characteristic of many of these models is the monotonic nature of productivity growth over time. In the absence of exogenous shocks, productivity growth is increasing over time, often due to population based scale effects in technology creation and, in some models, to increasing investment in human capital. A partial, but relevant, sample of this literature is discussed below.

An early and important paper in this literature is Kremer (1993) who presents a model of growth over the very long run. Key assumptions in the model are that productivity growth is an increasing function of the population level, and that there is a Malthusian-style constraint on population growth so that population growth depends (in the long run) on productivity growth. These assumptions imply that the world population growth rate is a function of the world population level, which fits the historical evidence, at least up until the demographic transition. Absence exogenous decreases in population levels (wars, plagues, etc.) productivity growth will be monotonically increasing over time, leading eventually to the high growth of the modern era.
The model of Galor and Weil (2000) has been very influential. The model captures the endogenous transitions between three growth ‘regimes’: the Malthusian regime, the post-Malthusian regime, and the Modern regime. In the Malthusian regime, population and productivity growth are low, income per capita is relatively stagnant, and higher income induces faster population growth. Productivity growth slowly accelerates over time as population levels increase. Eventually productivity growth is fast enough to permit significant growth in income per capita, leading to (relatively) high population growth rates, which characterizes the post-Malthusian regime. As productivity growth continues to increase, the return to human capital investments eventually becomes high enough so that parents focus more on the quality rather than the quantity of their offspring, leading to the demographic transition and the Modern growth regime. In the Modern growth regime high productivity growth spurs human capital accumulation, which lowers population growth and fuels even faster productivity and income growth.

This was one of the first models that could endogenously account for both the transition from millennia of relative stagnation in income per capita to sustained growth and for the demographic transition. However, since knowledge and economic productivity are not treated separately in this model, there can be no separate role for the Scientific Revolution to play. Also, the assumption that more investment in human capital leads to faster technological growth implicitly assumes that there exists an economically useful body of knowledge that is ready to be installed in the new generation. With a growing body of scientific knowledge this is a reasonable assumption to make. In the absence of a substantial, and for the long run growing, body of scientific knowledge there may not exist a clear relationship between spending more resources on education and increases in economic productivity. In terms of impact on technology growth, there is a world of difference for example, between training in electrical engineering and the study of Plato’s dialogues or Confucius’ Analects. That is to say, there were probably limited returns to increasing human capital investment in earlier societies, since there did not exist a sizeable body of scientific knowledge to draw upon (at least relative to 18th and 19th century Europe). These limitations can be incorporated in long run growth models, such as this paper’s,

---

8 For a comprehensive review of this model and related models, see Galor (2005).
9 This is not to say that there was no scientific knowledge existent in pre-modern societies, just that the amount of knowledge was fairly limited by modern standards. For example, it may be argued that with regards to Classical Mediterranean civilizations there was substantial scientific knowledge to be gained from education. Although this
by tying the accumulation of technologically productive human capital to scientific knowledge. In the long run, growth in scientific knowledge can be a limiting factor for growth in technologically productive human capital.

Like the above models, the calibrated model in Jones (2001) incorporates (weak) scale effects in technology growth, leading over time to rising productivity growth. The model does have the novel feature that institutional improvements in protecting the returns to innovation provide a mechanism for the take-off in growth that has occurred over the last couple of centuries. The timing, but not the inevitability, of an Industrial Revolution is affected by these institutional improvements.

The model of this paper differs from the above papers in two significant ways. First productivity growth is not monotonically increasing, although as will be seen in section 3 there is a sense in which ‘average’ productivity growth can be increasing over time, at least until the last accessible major innovation is discovered. Rather than having productivity growth increasing steadily over time (in the absence of exogenous shocks), productivity growth will be higher following the arrival of a major innovation, and then decrease over time until the next major innovation occurs. Second, it distinguishes between productivity growth and growth in scientific knowledge, allowing for a potentially separate treatment of the Industrial Revolution and the Scientific Revolution. Substantial productivity growth can occur without the equivalent of modern science, but in the very long run growth in scientific knowledge is necessary for sustained productivity growth.

More recently, O’Rourke et al (2007) present a model of the transition to modern growth that distinguishes between productivity growth (rising from applied knowledge) and growth in scientific (or Baconian) knowledge. Their model can account for the historical fact that technological change was unskilled biased during the early Industrial Revolution and generally skilled biased since. Growth in scientific knowledge is endogenous, depending positively on the amount of skilled workers in the economy, and is necessary for sustained growth. As long as there are skilled workers in the economy, growth in scientific knowledge will be positive, and the take-off to sustained modern growth is inevitable. This contrasts with this paper’s model, in

---

may be true in terms of mathematics, especially geometry, and to some extent astronomy, there is a large qualitative difference between the science and mathematics of Europe in the 19th century and that of Antiquity. In fact, much of the ‘knowledge’ of Antiquity had to be unlearned in order for modern science to emerge (for example, the physics and biology of Aristotle, the earth-centered astronomy of Ptolemy).

10 See also Jones (2005) for more on growth over the very long run and scale effects.
which something like the first Industrial Revolution could occur without necessarily transitioning to sustained growth.

### 3. A Simple Model

The following is a simple model that captures how growth in scientific knowledge can impact productivity, population, output, and human capital. The model is characterized by four principal equations, along with additional assumptions on the nature and arrival of major innovations and appropriate parameter restrictions. Initial values of the variables are assumed to be greater than zero.

#### A. Assumptions

It is assumed that growth in scientific knowledge is exogenous,

\[
\frac{\dot{\Gamma}_t}{\Gamma_t} = g_t
\]

where \( \Gamma_t \) is a measure of the body of scientific knowledge at time \( t \). Throughout the model \( g_x \) will denote the growth rate of \( X \). Some scientific knowledge is assumed to always exist, in the form of knowledge of how, to some degree, the natural world operates. For example, agriculture requires knowledge of the seasons and/or weather patterns, soil fertility, etc. Some major innovations are only accessible, however, with certain levels of knowledge corresponding to modern, formal science. Potential modifications of this assumption are discussed in section 5.

Productivity growth is determined by

\[
g_A = L^{\lambda} h^\sigma \left( \frac{A}{\Omega} \right)^{-\mu} \quad \Omega_0 = A_0 = 1
\]

where \( A \) is economic productivity, \( L \) is the size of the population, \( h \) is the quality of human capital embodied in \( L \), \( \Omega \) is a variable associated with the last major innovation discovery, and \( \lambda \), \( \sigma \), and \( \mu \) are positive constants. Holding \( \Omega \) constant, this gives the growth rate of productivity due to incremental innovations and captures the ‘normal’ or ‘routine’ innovation
process (whereas the arrival of a major innovation would signify a significant/general purpose technology discovery). Productivity growth is thus increasing in the size of the population and the quality of human capital. Barring the arrival of a new major innovation, productivity growth is decreasing in the level of productivity, representing a ‘fishing-out’ effect so that new incremental innovations become increasingly difficult to discover.

Population growth depends on productivity growth and human capital growth,

\[ A3) \quad g_L = \min[\beta_1 g_A, \bar{g}_L] - \beta_2 g_h \]

where \( \bar{g}_L \) is the maximum feasible growth rate of the population and \( \beta_1 \) and \( \beta_2 \) are positive constants. This equation determines the growth rate of the population and represents a long run equilibrium condition. Population growth depends positively on productivity growth due to some Malthusian type of constraint. In the long run, it is growth in economic productivity that allows the population to expand. Introducing a maximum for the population growth rate is plausible, since there are biological limits to how fast population can grow (for example, in the absence of immigration, population growth could never be 20%). It also serves to break the linearity of equation A3, which will allow for a demographic transition to occur. Population growth depends negatively on the growth rate of the quality of human capital. This captures the trade-off parents face in fertility decisions between the number of offspring and the amount of resources per child devoted to quality in terms of education/human capital. A more detailed description of population growth is analyzed in the appendix\(^{11}\).

Growth in the quality of human capital is dependent on growth in scientific knowledge,

\[ A4) \quad g_h = g_r \]

\(^{11}\) See Galor and Weil (2000) for a model of the demographic transition based on parents’ optimally choosing the quantity and quality (in terms of human capital) of their offspring. Having the population growth rate determined by such optimization behavior, along with a more detailed description of the economic incentives to acquire human capital and discover innovations, would not change in the model the basic interaction between science and growth which is the focus of this paper. For this reason, in interests of simplicity, the model is presented in a ‘reduced-form’ fashion.
This equation, as with assumption 3, represents a long run equilibrium condition. The growth rate of the quality of the human capital embodied in the population is equal to the growth rate of scientific knowledge, so that in the long run growth in human capital is limited by growth in scientific knowledge. A somewhat more detailed description of human capital growth is analyzed in the appendix.

The production function takes the following simple form,

\[ Y_t = A_t L_t^\alpha \]

where \( Y_t \) is aggregate output, which is produced by combining labor and technology. Production thus exhibits decreasing returns to labor, which represents the presence of some fixed, but renewable, factor like land that is essential for production.

The next three assumptions describe the nature and arrival of major innovations.

\( A6) \quad \Omega_t = A_t \)

where \( t' \) is the time of the last major innovation discovery. This states that the arrival of a new major innovation ‘sets-back’ the fishing-out effect. Thus at the time of arrival of a major innovation, the term \( \left( \frac{A}{\Omega_t} \right)^{-\mu} \) in equation A2 is equal to one, and then gradually goes to zero as \( A \) increases (until the next major innovation arrives)\(^{12} \). For simplicity it is assumed that for a given population level and quality of human capital embodied in that population the ‘size’ of each major innovation, that is the impact on productivity growth, is always the same, although in principle of course this may not be true.

\( A7) \quad \text{The number of accessible (already discovered and yet to be discovered) major innovations is equal to } \omega(\Gamma) , \text{ where:} \)

---

\(^{12}\) This formulation assumes that productivity growth makes an immediate and discrete jump upwards when a major innovation is discovered. This is obviously a simplification, since it takes time for innovations to be implemented and to spread. This formulation does, however, capture the essential point that these innovations play themselves out over time leading to lower productivity growth, in the absence of growth in scientific knowledge or new major innovations.
i) \( \omega(\Gamma) \rightarrow \mathbb{N} \) (the natural numbers)

ii) For every \( \Gamma \) there exists a \( \Gamma' > \Gamma \) such that \( \omega(\Gamma') = \omega(\Gamma) + 1 \)

The ability to discover major innovations is assumed to be a function of the current level of scientific knowledge, and for a given level of knowledge, the number of discoverable major innovations is finite. If scientific knowledge is not sufficiently advanced, then some major innovations are simply not accessible, i.e. able to be discovered/implemented. For example, a mathematical understanding of electromagnetism is necessary for the widespread generation, transmission, and use of electricity. Note that A7 also implies that if knowledge advances far enough, there are always major innovations still to be discovered, which seems to be a reasonable assumption to make at least for the foreseeable future.

A8) The arrival of major innovations follows a random process with the property that for a given \( \Gamma \), if there are still undiscovered accessible major innovations, then the major innovation arrival rate is increasing in \( L \) and \( h \).

This assumes that more people and higher quality of human capital increases the likelihood of the discovery of a major innovation, as long as some remain accessible (given the current level of scientific knowledge) and undiscovered.

B. Productivity growth in the steady state

In the absence of the arrival of major innovations, how does this system evolve over time? First, we analyze what happens to the growth rate of productivity by examining the growth rate of the growth rate of productivity.

1) growth rate of \( g_A = \lambda g_L + \sigma g_h - \mu g_A \)

\[
= \lambda \min[\beta_1 g_A, g_L] - \lambda \beta_2 g_h + \sigma g_h - \mu g_A \\
= \lambda \min[\beta_1 g_A, g_L] - \mu g_A + [\sigma - \lambda \beta_2] g_I
\]

We make the following additional assumption so that in the absence of growth in scientific knowledge (and new major innovations), productivity growth decreases over time even as the population grows:
A9) $\mu > \lambda \beta_1$

Growth in scientific knowledge has two effects on productivity growth due to incremental innovations: it raises the quality of the human capital embodied in the population, and it causes the population growth rate to decrease (for a given growth rate of productivity) as parents substitute quality for quantity. In order that such a rise in human capital outweighs the effect of slower population growth, it is assumed that:

A10) $\sigma > \lambda \beta_2$

Going back to equation #1, under these two assumptions we have that:

$$\mu g_A - \lambda \min[\beta_i g_A, \overline{g_L}] > 0 \quad \text{and} \quad [\sigma - \lambda \beta_2] g_r > 0$$

which gives us figure 1, where the vertical distance between the $[\sigma - \lambda \beta_2] g_r$ line and the $\mu g_A - \lambda \min[\beta_i g_A, \overline{g_L}]$ line gives the growth rate of $g_A$.

Thus over time the growth rate of productivity will converge to the steady state growth rate (here drawn arbitrarily to the left of the ‘kink’ in $\mu g_A - \lambda \min[\beta_i g_A, \overline{g_L}]$ and with $g_r > 0$).

This steady state growth rate will be:

2) $g_A^* = \frac{(\sigma - \lambda \beta_2) g_r}{\mu - \lambda \beta_1}$ if $g_r < \frac{\overline{g_L}(\mu - \lambda \beta_1)}{\beta_1(\sigma - \lambda \beta_2)}$ (left of kink)

$$= \frac{(\sigma - \lambda \beta_2) g_r + \overline{\lambda g_L}}{\mu} \quad \text{if} \quad g_r \geq \frac{\overline{g_L}(\mu - \lambda \beta_1)}{\beta_1(\sigma - \lambda \beta_2)} \quad \text{(right of kink)}$$

so that the growth rate of productivity will be increasing in the growth rate of scientific knowledge.
C. Population growth in the steady state

The population growth rate in the steady state will be:

\[
3) \quad \frac{g^*_L}{\mu} - \frac{\beta g^*_A}{\lambda \beta} = \left(\frac{\sigma \beta - \mu \beta}{\mu - \lambda \beta}\right) g^*_L \quad \text{if} \quad g^*_A < \frac{g^*_L}{\beta} \\
= \frac{-g^*_L}{\beta} \quad \text{if} \quad g^*_A \geq \frac{g^*_L}{\beta}
\]

The following assumption is made so that population growth can be (but is not necessarily) positive in the steady state, if the growth rate of scientific knowledge is positive:

A11) \( \sigma \beta > \mu \beta \)

(note that A11 and A9 together imply A10)

D. Income per capita growth

Income per capita is \( y_t = \frac{Y_t}{L_t} = A_t L^{\alpha - 1} \). The growth rate of income per capita will be:

\[
4) \quad g_y = g_A - (1 - \alpha) g_L = g_A - (1 - \alpha) \left(\min[\beta, g_A, g^*_L] - \beta g_h\right)
\]

Suppose that in the absence of growth in human capital and the resulting diversion of resources away from quantity to quality, there is a Malthusian dynamic operating that keeps per capita income constant (as long as population growth remains below \( g^*_L \)). More formally, assume that:

A12) \( \beta = \frac{1}{1 - \alpha} \)

This implies that when productivity increases, population growth responds by growing just fast enough to keep income per capita constant; higher productivity shows up in population levels rather than income. Then the growth rate of per capita income would become:
\[ \frac{\dot{y}_t}{y_t} = (1 - \alpha) \beta g \Gamma \]

if \( g_A < \frac{g_L}{\beta} \)

\[ = g_A - (1 - \alpha) g_L + (1 - \alpha) \beta g \Gamma \]

if \( g_A \geq \frac{g_L}{\beta} \)

Note that if \( g_A \geq \frac{g_L}{\beta} \) then \( g_A - (1 - \alpha) g_L > 0 \) under assumption A12. Thus if productivity growth is high enough, income per capita growth is positive even in the absence of growth in scientific knowledge. A more general treatment of how population growth responds to income can be found in the appendix.

**E. Arrival of major innovations**

The arrival of a major innovation does not change the steady state growth rate of productivity, instead it temporarily raises productivity growth above the steady state rate, to which it then falls back over time, as illustrated in figure 2. The ‘size’ of the initial impact of a major innovation on productivity growth is given by:

\[ g_A = L^\lambda_i h_i^\sigma (\frac{A}{A_i})^{-\mu} = L^\lambda_i h_i^\sigma \]

This means that the impact of each major innovation will be greater the larger the population is and the greater the quality of human capital. If there are more people to work with, and improve upon, a general purpose type of technology (or the quality of their knowledge is higher), then productivity growth due to incremental innovations will be higher. This would also imply that as the population grows following the arrival of major innovations, both the arrival rate and the initial impact of major innovations will increase over time, at least until no accessible major innovations remain to be discovered.

**4. Applications**

We can now use this simple model to capture what long run productivity and population growth would look like if there was no growth in scientific knowledge, as well as the emergence of modern growth that is a result of an acceleration in the growth of scientific knowledge.
A. Long run productivity and population growth with $g_t = 0$

For ease of exhibition, suppose that $g_A < \frac{g_L}{\beta_1} \quad \forall t$, so that $g_L = \beta_1 g_A$ and the growth rate of $g_A = -[\mu - \lambda \beta_1] g_A$ (holding $\Omega$ constant). Productivity growth over time will follow a pattern similar to that exhibited in figure 3. In this figure it is assumed that at time zero there are three accessible major innovations given the (fixed) level of scientific knowledge, where $t_1$, $t_2$, and $t_3$ represent the times of discovery of these major innovations. The major innovation discovered at time $t_3$ is the last one accessible given the fixed level of scientific knowledge. Since population growth is proportional to productivity growth, it will also follow the same pattern.

As figure 3 suggests, a trend of average productivity (and population) growth could be rising over time, at least until the last accessible major innovation is discovered. Such a rising trend would reflect that the impact on productivity and the arrival rate of major innovations are both increasing over time due to rising population levels. With the arrival of the last major innovation (an ‘Industrial Revolution?’) initially there is an increase in productivity growth, but as useful innovations related to this major innovation are ‘fished out’ productivity growth falls eventually back to zero. The population growth rate increases temporarily, but also eventually returns to zero. Under assumption A12 per capita income growth will not increase (from zero) unless the size of the initial productivity shift moves productivity growth to the right of the kink. If that does occur then income per capita growth can be positive until productivity growth moves back to the left of the kink. The growth rate of human capital quality remains zero throughout.

B. An acceleration in the growth rate of scientific knowledge, an Industrial Revolution, and a Demographic Transition

Suppose that the growth rate of scientific knowledge accelerates from zero to $g_{\Gamma_1} > 0$. This causes the growth rates of human capital, productivity, and population to increase, which, by assumption A8 makes the arrival of a major innovation (the Industrial Revolution) more likely. Suppose that this major innovation arrives, as depicted in figure 4, and concurrently, the growth rate of scientific knowledge continues to accelerate. In this case, productivity growth does not diminish following the Industrial Revolution, but instead accelerates even more. The population
growth rate increases up to the kink, then begins to decrease, and the per capita income growth rate increases as $g_t$ increases, and receives an additional ‘boost’ once the kink is passed. The growth rate of human capital increases as $g_t$ increases. Finally, since scientific knowledge is increasing over time, new major innovations will become accessible, so that productivity growth can temporarily rise above the new steady state (electrical power, IT revolution can be examples). In fact we would expect from the model that major innovations should arrive more frequently, and possibly, depending on the relative ‘size’ of each major innovation, have a greater impact on productivity.

In the model, therefore, the Industrial Revolution of the late 18th and early 19th centuries could be treated as similar to other ‘Productivity Revolutions’ of the past\textsuperscript{13}. Perhaps it was not the Industrial Revolution per se that was the significant break with the past. Instead it could be argued that it was the continuation of productivity growth after the initial Industrial Revolution, and the increasing reliance on scientific knowledge to fuel that growth, that was the regime shift. As Mokyr (2002, p31) argues “The true question of the Industrial Revolution is not why it took place at all but why it was sustained beyond, say, 1820.” This approach would be more consistent with the view that the early years of the Industrial Revolution marked no abrupt or unprecedented change in productivity growth (see for example Clark (2001, 2003) for an exposition of this view).

5. Towards Endogenous Growth in Scientific Knowledge

Although there may always be elements of the emergence of modern science that cannot be captured in a deterministic way within a model, there are potentially important feedback effects between demography, the economy, and science that should be considered. Given that a culture of scientific inquiry and discovery already exists, population levels, human capital, and technology can all play a role in the pace of scientific advancement. A larger population could mean more researchers in basic science and a greater likelihood of an equivalent to a Darwin or Einstein emerging. Since the quality of human capital in this model reflects the amount of scientific knowledge embodied in the labor force, more human capital would make scientific

\textsuperscript{13} Although the impact of the Industrial Revolution could be greater and diffuse more quickly than previous productivity revolutions due to a relatively larger size of population at the time and the increasingly global nature of the world economy.
discoveries more likely. And the process of scientific discovery is itself increasingly dependent on technology, computing power and the construction of technologically sophisticated particle accelerators being two examples. In order for a balanced growth path to exist it will be assumed that for fixed levels of population, human capital, and technology, the growth rate of scientific knowledge is decreasing in the level of scientific knowledge: new knowledge becomes increasingly difficult to acquire if more resources are not devoted to scientific discovery\textsuperscript{14}.

Let $\Pi(L, A) \rightarrow \{0,1\}$ be a random variable that represents the presence ($\Pi = 1$) or lack ($\Pi = 0$) of a culture of formal, quantitative science. A larger population can increase the likelihood of persons like a Galileo or Newton being born who could contribute to the beginnings of a scientific culture. Another factor that could affect the likelihood of a scientific culture emerging is economic productivity $A$, since a relatively wealthy and technologically sophisticated (both by pre-modern standards) civilization is probably necessary to support scientific research. For simplicity it is assumed that once a scientific culture is established it is irreversible, $\Pi_t = 1 \Rightarrow \Pi_{t'} = 1$ for all $t' > t$\textsuperscript{15}. The growth rate of scientific knowledge, given that a scientific culture already exists, is assumed to be a function of the population level, the quality of the human capital embodied in the population, the technology level as captured in the productivity parameter, and the current amount of scientific knowledge. Assumption A1 could then be replaced with:

\[ A1') \quad g_r = \gamma(L, h, A, \Gamma) \ast \Pi(L, A) \quad \gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0, \gamma_4 < 0 \]

Under these assumptions, the likelihood of a scientific revolution occurring would increase over time as population levels and productivity rise following the arrival of major innovations. If and when a scientific culture does emerge (and not necessarily preceding or coinciding with the equivalent of the first Industrial Revolution, although the existence of a scientific culture would make an Industrial Revolution more likely to occur), then, with the appropriate assumptions on the function $\gamma$, the growth rates of productivity and knowledge will increase until a balanced growth path is reached. On the balanced growth path the growth rates of $h, A, \text{ and } \Gamma$ will all be

\textsuperscript{14} Without this assumption, in the model both productivity growth and growth in scientific knowledge would be permanently accelerating over time.

\textsuperscript{15} This assumption is rather strong from a historical perspective. Growth in scientific knowledge is not necessarily self-sustaining and knowledge can be, and has been, lost or forgotten. For example, the beginnings of a scientific culture in Classical Greek civilization did not lead to continuous growth in scientific knowledge.
constant and positive, implying that the growth rates of $L$ and $\gamma$ be constant as well. The appendix contains a characterization of this dynamical system for a specific functional form for $\gamma$.

6. Conclusion

Increases in scientific knowledge are critical, indeed essential, to sustained economic growth. Models of growth over the very long run could be enriched by including explicitly growth in scientific knowledge as distinct from (but related to) growth in technology/productivity. The contribution of this paper is to present a simple model that examines how productivity, population, the quality of human capital, and output can be affected by growth in scientific knowledge. In the model productivity growth in the very long run is dependent on advances in science. This long run dependence does not preclude the possibility of significant and recurrent productivity growth occurring during the pre-modern era. It does, however, place limits on the growth potential of previous societies and highlights the uniqueness of Western European civilization at the dawn of the modern era.
References


Appendix

A. Population and human capital growth

Assumptions A3 and A4 represent equilibrium conditions. A more detailed description of how population growth and growth in the quality of human capital evolve over relatively short periods of time could be obtained by replacing A3 and A4 with the following:

\[
\begin{align*}
A3') \quad g_L &= n(y, h, g_h) & \text{with} & \quad n_1 > 0, \quad n_2 < 0, \quad n_3 < 0 \\
A4') \quad g_h &= f\left(\frac{\Gamma}{h}\right) & \text{with} & \quad f' > 0, \quad f(1) = 0, \quad \lim_{n \to \infty} f' > 0
\end{align*}
\]

Population growth responds positively to income and negatively to the level of human capital and the growth rate of human capital, since maintaining and increasing human capital potentially diverts resources away from child quantity. Assumption A4' states that the growth rate of human capital increases as the gap between the level of scientific knowledge and the level of human capital increases.

Given that all the rest of the assumptions from section 2A are the same, we wish to characterize the balanced growth path for this economy where all variables grow at constant rates, in the absence of the arrival of a major innovation.

If \( g_\Gamma \) is constant, then \( g_h \) will converge to \( g_\Gamma \), under the assumptions made on the function \( f\left(\frac{\Gamma}{h}\right) \) as is clearly seen the figure 5. Given \( g_h = g_\Gamma \) is constant, what conditions must hold so that \( g_A \) and \( g_L \) (and hence \( g_y \), as well) are constant over time? From assumption A2 we have that the growth rate of \( g_A = \lambda g_L + \sigma g_h - \mu g_A \), so that \( g_A = 0 \) if the following equation is satisfied:

\[
g_A = \frac{\lambda}{\mu} g_L + \frac{\sigma}{\mu} g_\Gamma \quad \Rightarrow \quad [g_A = 0]
\]
The population growth rate is constant, \( g_L = 0 \), if \( n_1 \dot{y} + n_2 \dot{h} + n_3 \dot{g}_h = 0 \). Given \( g_h = g_\Gamma \) is constant, this condition can be written as \( g_y = -(\frac{n_2}{n_1}) \dot{h} g_\Gamma \). This implies that \( g_y \) (and hence \( g_L \)) will only be constant over time if \(-\left(\frac{n_2}{n_1}\right) > 0\) is constant over time. It is therefore assumed that the function \( n(y, h, g_h) \) is such that \(-\left(\frac{n_2}{n_1}\right) = \bar{n} \) is constant if \( g_h \) is constant. For example, the function \( n = y^\beta h^{-\beta_2} (g_h + 1)^{-\beta_3} - \beta_4 \) would satisfy the properties of \( n \), and we would have that \( \bar{n} = \frac{\beta_2}{\beta_1} \) is constant. So given this additional assumption on the function \( n \), we have that the population growth rate is constant if \( g_y = \bar{n} g_\Gamma \), which becomes:

7) \( g_\Lambda = (1 - \alpha)g_L + \bar{n} g_\Gamma \) \hspace{1cm} [\( g_L = 0 \)]

Thus we have two equations in \( g_\Lambda \) and \( g_L \) that must be satisfied simultaneously (given \( g_h = g_\Gamma \) is constant) for the economy to be on the balanced growth path.

In order to have global stability in this dynamical system it is assumed that:

8) \( (1 - \alpha) > \frac{\lambda}{\mu} \)

and, given the above inequality, in order for the population growth rate to be positive along the balanced growth path it is also assumed that:

9) \( \frac{\sigma}{\mu} > \bar{n} \)

Equations 6 and 7, along with parameter assumptions 8 and 9 and \( g_\Gamma > 0 \) gives the phase diagram depicted in figure 6. The arrival of a major innovation temporarily raises productivity growth above its steady state level, to which it then converges back to. Note that if \( g_\Gamma \) increases, say from \( g_{\Gamma_1} \) to \( g_{\Gamma_2} \), then, since \( g_h \) doesn’t immediately increase by the same amount (instead
converges from below to the new level of $g_r$, the shifting of the two curves wouldn’t be immediately to the amounts implied by $\frac{\sigma}{\mu} g_{r2}$ and $\bar{n} g_{r2}$.

Using equations 6 and 7, we can solve for the balanced growth path values of $g_L$, $g_A$, and $g_y$:

10) $g_L^* = -\frac{\sigma - \bar{n}}{\mu \left(1 - \alpha - \frac{\lambda}{\mu}\right)} g_r$

11) $g_A^* = \frac{(1 - \alpha) \frac{\sigma}{\mu} - \left(\frac{\lambda}{\mu}\right) \bar{n}}{1 - \alpha - \frac{\lambda}{\mu}} g_r$

12) $g_y = \bar{n} g_r$

A demographic transition could be modeled as an decrease in $n_1$ (the effect of income on population growth) which would, other things being equal, increase $\bar{n}$. This decrease in $n_1$ could occur as income grows above as certain level, or it could occur as $g_h$ increases.

If $g_r = 0$, then the phase diagram becomes that of figure 7, where the balance growth path levels of $g_A$, $g_L$, and $g_y$ are all zero.

B. Growth in scientific knowledge

Suppose that $\Pi = 1$ and that the function $\gamma$ takes the simple form listed below:

$$g_r = L^\phi_1 h^\phi_2 A^\phi_3 \Gamma^{-\phi_4}$$

where $\phi_1, \phi_3, \phi_4$, and $\phi_4$ are positive constants.
From this, and (the original) assumptions A3 and A4, it follows that that the growth rate of $g_r$ will be constant (in the absence of the arrival of a major innovation) if:

13) $g_A = \frac{\phi_4 + \phi_1 \beta_2 - \phi_2}{\phi_1 \beta_1 + \phi_3} g_r$ if $g_A < \frac{g_L}{\beta_1}$

$$= \frac{\phi_4 + \phi_1 \beta_2 - \phi_2}{\phi_3} g_r - \frac{\phi_1 g_L}{\phi_3}$$ if $g_A \geq \frac{g_L}{\beta_1}$

The growth rate of $g_A$ will be constant if equation 2 is satisfied:

2) $g_A = \frac{\sigma - \lambda \beta_2}{\mu - \lambda \beta_1} g_r$ if $g_A < \frac{g_L}{\beta_1}$

$$= \frac{\sigma - \lambda \beta_2}{\mu} g_r + \frac{\lambda g_L}{\mu}$$ if $g_A \geq \frac{g_L}{\beta_1}$

This dynamical system will be globally stable if:

14) $\frac{\sigma - \lambda \beta_2}{\mu - \lambda \beta_1} > \frac{\phi_4 + \phi_1 \beta_2 - \phi_2}{\phi_1 \beta_1 + \phi_3}$

and

15) $\frac{\phi_4 + \phi_1 \beta_2 - \phi_2}{\phi_3} > \frac{\sigma - \lambda \beta_2}{\mu}$

Figure 8 illustrates the phase diagram for this system. The system will converge to $(g_r^*, g_A^*)$. At this steady state the growth rates of the population, human capital, and income will be constant as well. The arrival of a major innovation temporarily raises productivity growth above its steady state level, to which it then converges back to.
Figure 1

\[ \mu g_A - \lambda \min[\beta_1 g_A, g_L] \]

\[ [\sigma - \lambda \beta_2] g_T \]

Figure 2

\[ \mu g_A - \lambda \min[\beta_1 g_A, g_L] \]

\[ [\sigma - \lambda \beta_2] g_T \]
Figure 7

\[ g_A = 0 \]

\[ g_L = 0 \]

Figure 8

\[ g_A = 0 \]

\[ g_L = 0 \]